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SOME ALTERNATE CHARACTERIZATIONS OF RELIABILITY DOMINATION

F.T. BOESCH,* A. SATYANARAYANA, AND C.L. SUFFEL†
Stevens Institute of Technology
Hoboken, New Jersey 07030

An important problem in reliability theory is to determine the reliability of a system from the reliability of its components. If E is a finite set of components, then certain subsets of E are prescribed to be the operating states of the system. A formation is any collection F of minimal operating states whose union is E. Reliability domination is defined as the total number of odd cardinality formations minus the total number of even cardinality formations. The purpose of this paper is to establish some new results concerning reliability domination. In the special case where the system can be identified with a graph or digraph, these new results lead to some new graph-theoretic properties and to simple proofs of certain known theorems. The pertinent graph-theoretic properties include spanning trees, acyclic orientations, Whitney's broken cycles, and Tutte's internal activity associated with the chromatic polynomial.

1. INTRODUCTION

There have recently been several important advances in system reliability theory. One of the major issues in reliability theory is the determination of the reliability of a given system from the reliabilities of its components. System reliability includes a variety of network reliability problems that occur when the

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that by a *graph* we mean what Harary calls a *pseudograph*. Furthermore, in what follows, by *domination* of a graph or a system we mean the *reliability domination*.

Let E be a finite set and $\mathbf{P}(E)$ be the power set of E. A nonempty subset $\mathbf{C} \subseteq \mathbf{P}(E)$ is called a *clutter* on E if for any two elements $C_1 \in \mathbf{C}$ and $C_2 \in \mathbf{C}$, whenever $C_1 \subseteq C_2$, then $C_1 = C_2$. The pair $(E,\mathbf{C})$ will be referred to as a *system* and the system is *coherent* if each element of E is contained in some element of $\mathbf{C}$. A subset $A \subseteq E$ is called an *operating state* of the system $(E,\mathbf{C})$ if A contains an element of $\mathbf{C}$. Let $\Theta(E,\mathbf{C}) = \{A \subseteq E\colon C \subseteq A$ for some $C \in \mathbf{C}\}$ be the collection of all operating states of the system $(E,\mathbf{C})$. A *formation* F of a system $(E,\mathbf{C})$ is a subset of $\mathbf{C}$ with the property that $\bigcup_{C\in F} C = E$. The formation F is *odd* or *even* depending on whether its cardinality is odd or even, respectively. The *signed domination* $d(E,\mathbf{C})$ of a system $(E,\mathbf{C})$ is defined to be the number of odd formations minus the number of even formations of $(E,\mathbf{C})$.

The notion of the signed domination was introduced in [13] in the context of reliability analysis of directed networks. Suppose $G = (V,E)$ is a digraph and $K \subseteq V$ is a specified subset of points of G such that $s \in K$. A subdigraph T of G is a *rooted tree*, rooted at s, if in T the indeg(s) = 0 while the other points of T have indegree = 1. A *K-tree*, rooted at s, is a rooted tree, rooted at s, such that (i) $s \in K$, (ii) every point of K lies on the tree, and (iii) every point with outdegree = 0 is a K-point. Clearly, a K-tree rooted at s of a digraph G constitutes a minimal subgraph G with the property that there is a directed path from s to each point of K. The subgraph is minimal, in the sense that deletion of any edge from it results in the event that not all points in K can be reached from s.

A digraph $G = (V,E)$ with $K \subseteq V$ and $s \in K$ is called a *K-digraph* if every edge of G lies in some K-tree, rooted at s, of G. Let $\mathfrak{Z}_K(G)$ be the collection of all the K-trees, rooted at s, of G. Clearly, $\mathfrak{Z}_K(G)$ constitutes a clutter on E. Furthermore, the system $[E,\mathfrak{Z}_K(G)]$ is coherent if and only if G is a K-graph. A *formation* F of G is a collection of K-trees, rooted at s, whose union constitutes the set of edges E of G. A formation F is *odd* or *even* depending on whether F contains an odd or even number of trees, respectively. The *signed domination* of a digraph $G = (V,E)$, with respect to a given subset $K \subseteq V$ and $s \in K$, is the number of odd minus the number of even formations of G. In this instance, we write $d_K(G)$ instead of $d[E,\mathfrak{Z}_K(G)]$. The absolute value of $d_K(G)$ will be noted by $D_K(G)$.

The notions of K-trees, K-digraphs, formations, and the signed domination are applicable to undirected graphs as well. Suppose $G = (V,E)$ is an undirected graph and $K \subseteq V$. A K-tree of G is a tree of G containing all points of K such that every leaf of the tree belongs to K. The notions of K-graph, formation, and the signed domination are similarly defined.

The invariant $d_K(G)$ has been used in the following directed network reliability problem: We are given a directed network $G = (V,E)$ with $K \subseteq V$ and $s \in K$. The elements (edges or points) of G, at a given instant of time, are in one of two states, either failed or functioning. A point u is said to be able to com-

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We again refer to the digraph example of Figure 1. Here $E = \{x_1, x_2, x_3, x_4, x_5\}$, $\mathbf{C} = \{\{x_1, x_4\}, \{x_1, x_3, x_5\}, \{x_2, x_5\}\}$, and $x = x_3$. Then, $\mathbf{C}_{-x} = \{\{x_1, x_4\}, \{x_2, x_5\}\}$ and $\mathbf{C}_{+x} = \{\{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_4\}\}$. Note that $\{E - \{x\}, \mathbf{C}_{-x}\}$ is the system associated with the communication from s to t in $G - x$. On the other hand, $\{E - \{x\}, \mathbf{C}_{+x}\}$ does not describe the same phenomenon in $G|x$; specifically, $G|x$ does not describe the behavior of G when x is functioning. Indeed, $\{x_2, x_4\}$ is an s to t path in $G|x$ but certainly does not represent a valid path in G when x_3 is functioning. Finally, note that $d(E,\mathbf{C}) = 1$, $d(E - \{x\}, \mathbf{C}_{-x}) = -1$, and $d(E - \{x\}, \mathbf{C}_{+x}) = 0$, so that $d(E,\mathbf{C}) = d(E - \{x\}, \mathbf{C}_{+x}) - d(E - \{x\}, \mathbf{C}_{-x})$ in our example. The fact that $d(E,\mathbf{C}) = d(E - \{x\}, \mathbf{C}_{+x}) - d(E - \{x\}, \mathbf{C}_{-x})$ holds for any coherent system $(E,\mathbf{C})$ has been proved in Barlow [2]. We note the following elementary consequence of the definition of $d_K(G)$.

Fact 2.1: *Let $G = (V, E)$ be a graph or a digraph and K a subset of V. If u is an isolated point of G which is not in K, then $d_K(G - u) = d_K(G)$.*

In view of the above fact it is henceforth assumed, unless stated otherwise, that the graphs or digraphs we deal with have no isolated points u such that

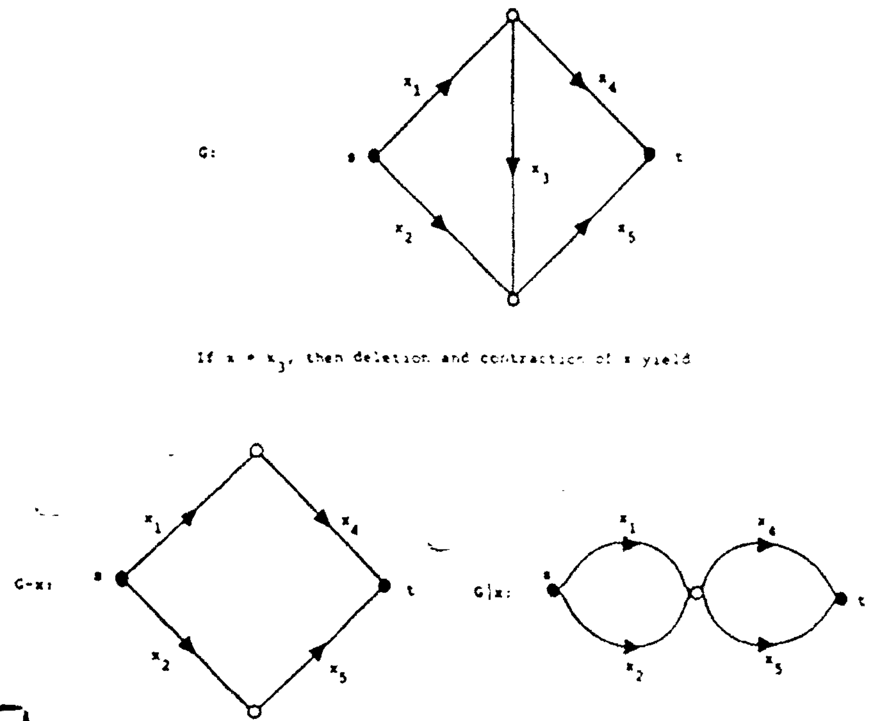


Figure 1. An example digraph G, $G - x$, and $G|x$.

$u \in V - K$. We conclude the preliminaries with two useful propositions, the proofs of which may be found in Satyanarayana and Chang [10].

PROPOSITION 2.1: *Let $G = (V,E)$ be an undirected graph such that $|V| > 1$, and $K \subseteq V$. Then G is not a K-graph if and only if one of the following holds:*

- (*i*) *G contains two connected components, each of which has an edge,*
- (*ii*) *G contains two connected components, each of which has a K-point,*
- (*iii*) *G is connected with $|K| < 2$,*
- (*iv*) *G has a cutpoint u such that $G - u$ has a component with no point from K,*
- (*v*) *G has a self-loop.*

PROPOSITION 2.2: *Suppose $C = (V,E)$ is an undirected graph, and $K \subseteq V$. Then $d_K(G) \neq 0$ if and only if G is a K-graph.*

3. DOMINATION AND SUBSTRUCTURES

Theorem 3.1 of this section, first proved by Huesby [5], provides a characterization of domination for systems. This result is used to establish some new results concerning domination of graphs and digraphs. This section also includes simple proofs of some of the well-known results.

Let $\mathbf{C}$ be a clutter on the set E, and let $\Theta(E,\mathbf{C})$ be the collection of all operating states S of the associated system. We can partition the set $\Theta(E,\mathbf{C})$ of operating states into two classes, depending upon the cardinality of the states, as follows:

$$\Theta_o(E,\mathbf{C}) = \{S \in \Theta(E,\mathbf{C}) \colon |S| \text{ is odd}\} \quad \text{and}$$

$$\Theta_e(E,\mathbf{C}) = \{S \in \Theta(E,\mathbf{C}) \colon |S| \text{ is even}\}.$$

THEOREM 3.1: *For any system $(E,\mathbf{C})$, $d(E,\mathbf{C}) = (-1)^{|E|}\,(|\Theta_e(E,\mathbf{C})| - |\Theta_o(E,\mathbf{C})|)$.*

Proof of Theorem 3.1 can be found in Huseby [5].

Note that Theorem 3.1 holds for any arbitrary clutter and, in particular, reaffirms the fact that the domination of a noncoherent system is zero. A special case of this theorem was discovered first by Rodriguez [7] for undirected graphs.

COROLLARY 3.1: *If $G = (V,E)$ is a graph or a digraph, $K \subseteq V$, and $s \in K$, then*

$$d_K(G) = (-1)^{|E|}(|\Theta_e(E,\mathfrak{J}_K(G))| - |\Theta_o(E,\mathfrak{J}_K(G))|),$$

where $\mathfrak{J}_K(G)$ is the set of K-trees of G, if G is a graph, and $\mathfrak{J}_K(G)$ is the set of K-trees, rooted at s, if G is a digraph.

A consequence of Theorem 3.1 is the following well-known result.

COROLLARY 3.2: (*Signed domination theorem*). *If* $(E,\mathbf{C})$ *is a system,* $x \in E$ *and* $\mathbf{C}_{+x}$ *and* $\mathbf{C}_{-x}$ *are the minors of* $\mathbf{C}$ *with respect to* x, *then* $d(E,\mathbf{C}) = d(E - \{x\}, \mathbf{C}_{+x}) - d(E - \{x\},\mathbf{C}_{-x})$.

PROOF: We first claim that $|\Theta_0(E,\mathbf{C})| = |\Theta_0(E - \{x\},\mathbf{C}_{-x})| + |\Theta_e(E - \{x\}, \mathbf{C}_{+x})|$, and $|\Theta_e(E,\mathbf{C})| = |\Theta_e(E - \{x\},\mathbf{C}_{-x})| + |\Theta_0(E - \{x\},\mathbf{C}_{+x})|$.

Indeed, if $S \subseteq E$ and $x \notin S$ then $S \in \Theta_0(E,\mathbf{C})$ if and only if there is an element $C \in \mathbf{C}$ which is a subset of S and consequently does not include x, that is, if and only if $S \in \Theta_0(E - \{x\},\mathbf{C}_{-x})$. If $x \in S$, then $S \in \Theta_0(E,\mathbf{C})$ if and only if there is an element $C \in \mathbf{C}$ such that $C \subseteq S$, that is, $C - \{x\} \subseteq S - \{x\}$. Hence, if $x \in S$, $S \in \Theta_0(E,\mathbf{C})$ if and only if $S - \{x\} \in \Theta_e(E - \{x\},\mathbf{C}_{+x})$ and the first equation is established. The second is proved in the same manner. We may therefore write

$$\begin{aligned}
d(E,\mathbf{C}) &= (-1)^{|E|}(|\Theta_e(E,\mathbf{C})| - |\Theta_0(E,\mathbf{C})|)\\
&= (-1)^{|E|}(|\Theta_e(E - \{x\},\mathbf{C}_{-x})| - |(\Theta_0(E - \{x\},\mathbf{C}_{-x})|\\
&\quad + |\Theta_0(E - \{x\},\mathbf{C}_{+x})| - |\Theta_e(E - \{x\},\mathbf{C}_{+x})|)\\
&= (-1)^{|E-\{x\}|}(|\Theta_e(E - \{x\},\mathbf{C}_{+x})| - |\Theta_0(E - \{x\},\mathbf{C}_{+x})|)\\
&\quad - (-1)^{|E-\{x\}|}(|\Theta_e(E - \{x\},\mathbf{C}_{-x})| - |\Theta_0(E - \{x\},\mathbf{C}_{-x})|)\\
&= d(E - \{x\},\mathbf{C}_{+x}) - d(E - \{x\},\mathbf{C}_{-x}),
\end{aligned}$$

where the first and the last equalities are justified by Theorem 3.1. ■

The first version of Corollary 3.2 was originally established by Satyanarayana [8] for the all-terminal domination of graphs. Satyanarayana and Chang [10] later extended it to the K-terminal domination of a graph. Subsequently, Barlow [2] has shown that the signed domination theorem holds for all coherent systems and this result was later extended to general clutters by Huseby [5,6].

If the system $(E,\mathbf{C})$ represents an undirected graph $G = (V,E)$ such that the clutter $\mathbf{C}$ is the collection of the K-trees of G, $K \subseteq V$, then it is easy to see that Corollary 3.2 reduces to the following:

COROLLARY 3.3: *If* $G = (V,E)$ *is an undirected graph,* $K \subseteq V$, *and* $x \in E$ *then* $d_K(G) = d_{K|x}(G|x) - d_K(G - x)$.

COROLLARY 3.4: *Let* $G = (V,E)$ *be an undirected graph, and* $K \subseteq V$ *be a nonempty subset. If* G *has* i *isolated points that are not in* K, *then* $D_K(G) = (-1)^{|E|-|V|+i+1}\, d_K(G)$.

PROOF: The proof is by induction on $|E|$. For the basis step, suppose that $|E| = 0$. If $|K| = 1$ then $d_K(G) = +1$. On the other hand if $|K| > 1$, then $d_K(G) = 0$. In either case the basis step is established.

For the inductive step, let G be a graph such that $|K| > 0$ and $|E| > 0$, and assume that the corollary holds for all graphs with fewer edges than G. Pick an

edge $x \in E$. If x is a self-loop, then $d_K(G) = 0$. Likewise, if x has an endpoint $u \in V - K$ such that u is a degree-one point of G, then $d_K(G) = 0$ and the result follows in either case. Hence, assume that the endpoints of x are distinct and if x has an endpoint whose degree is one in G then it is a K-point. By Corollary 3.3, we have $d_K(G) = d_{K|x}(G|x) - d_K(G - x)$. Note that the choice of x implies that the number of isolated points in $V - K$ is the same as that in $V|x - K|x$. Since $G - x$ and $G|x$ have fewer edges than G, using the induction hypothesis, we obtain

$$\begin{aligned} d_K(G) &= (-1)^{|E|-|V|+i+1} D_{K|x}(G|x) - (-1)^{|E|-|V|+i} D_K(G - x) \\ &= (-1)^{|E|-|V|+i+1}(D_{K|x}(G|x) + D_K(G - x)). \end{aligned}$$

Now if $|E| - |V| + i + 1$ is odd then, since $D_{K|x}(G|x)$ and $D_K(G - x)$ are non-negative, $d_K(G) \le 0$ and $d_K(G) = -D_K(G) = (-1)^{|E|-|V|+i+1}D_K(G)$. On the other hand, if $|E| - |V| + i + 1$ is even, then $d_K(G) = D_K(G) = (-1)^{|E|-|V|+i+1}D_K(G)$. ■

The following corollary is immediate from the proof of Corollary 3.4.

Corollary 3.5: *Let $G = (V,E)$ be an undirected graph, and $K \subseteq V$ be a nonempty subset. Suppose $x \in E$ is an edge such that (i) x is not a self-loop, and (ii) if x has an endpoint u in $V - K$, then u is not a degree-one point of G. Then $D_K(G) = D_{K|x}(G|x) + D_K(G - x)$.*

Corollaries 3.3 through 3.5 were proved in Satyanarayana and Chang [10] for undirected K-graphs. Since Corollary 3.5 does not hold for any arbitrary edge x, it is of interest to ask that if the equality $D_K(G) = D_{K|x}(G|x) + D_K(G - x)$ does not hold for some edge x of an undirected graph G, then what can we say about G and x? The following corollary answers this question.

Corollary 3.6: *Let $G = (V,E)$ be an undirected graph, and $K \subseteq V$. Then $D_K(G) \neq D_{K|x}(G) + D_K(G - x)$ if and only if (i) x is a self-loop and $G - x$ is a K-graph, or (ii) x is incident on a degree-one point $u \in V - K$ and $G - x$ is a K-graph.*

Proof: If either (i) or (ii) holds, then $D_K(G) = 0$ by Proposition 2.1. But $D_K(G - x) > 0$, by Proposition 2.2, since $G - x$ is a K-graph.

Conversely, suppose that neither (i) nor (ii) holds, then we show that $D_K(G) = D_{K|x}(G|x) + D_K(G - x)$. First, suppose that x is a self-loop. Then, as (i) does not hold, $G - x$ is not a K-graph. Since $G|x = G - x$ in this case, it follows that $D_K(G) = D_K(G - x) + D_{K|x}(G|x)$. Next, suppose that x is not a self-loop. If x is not incident on a degree-one point $u \in V - K$, then by Corollary 3.5, $D_K(G) = D_{K|x}(G|x) + D_K(G - x)$. Hence, assume that x is incident on a degree-one point $u \in V - K$. Then, as (ii) does not hold, $G - x$ is not a K-graph and $D_K(G - x) = 0$ by Proposition 2.2. But by Corollary 3.3, $d_K(G) = d_{K|x}(G|x) - d_K(G - x)$ and it follows that $D_K(G) = D_K(G - x) + D_{K|x}(G|x)$. ■

Our next corollary yields a characterization for the K-terminal domination of undirected graphs $G = (V,E)$ with $K \subseteq V$, in terms of certain spanning connected subgraphs of G. We denote by $c(G,K)$ the number of connected components of G which contain at least one point of K. Let $S_0(G,K)$ denote the number of spanning subgraphs S of G such that each S has an odd number of edges and $c(S,K) = 1$. Similarly, $S_e(G,K)$ is the number of spanning subgraphs with evenly many edges and $c(S,K) = 1$. The following is an immediate consequence of Theorem 3.1, and the facts that $|\Theta_0(E,\mathfrak{Z}_K(G))| = S_0(G,K)$ and $|\Theta_e(E,\mathfrak{Z}_K(G))| = S_e(G,K)$.

Corollary 3.7: *For any undirected graph $G = (V,E)$ with a specified subset $K \subseteq V$, $D_K(G) = |S_e(G,K) - S_0(G,K)|$.*

Corollary 3.7 specialized to the all-terminal domination yields the characterization of $D_V(G)$ in terms of spanning connected subgraphs of G. More specifically, $S_0(G, V)$ and $S_e(G, V)$ are the number of spanning connected subgraphs with odd and even number of edges, respectively. However, we can partition the spanning connected subgraphs of G into two classes based on the nullity of the subgraphs rather than the number of edges in the subgraph. Specifically, let $\Theta(G)$ denote the collection of spanning connected subgraphs of G. If $H \in \Theta(G)$, and $n(H)$, $e(H)$ denote the number of points and edges of H, respectively, then let $S_0(G) = |\{H \in \Theta(G): (e(H) - n(H) + 1) \text{ is odd}\}|$ and $S_e(G) = |\{H \in \Theta(G): (e(H) - n(H) + 1) \text{ is even}\}|$. It is clear that, in general, $S_0(G, V) \neq S_0(G)$ and $S_e(G, V) \neq S_e(G)$. The following corollary provides another characterization of $D_V(G)$ in terms of $S_0(G)$ and $S_e(G)$. It follows from the observation that either $S_0(G, V) = S_0(G)$ and $S_e(G, V) = S_e(G)$ or $S_0(G, V) = S_e(G)$ and $S_e(G, V) = S_0(G)$ together with an application of Corollaries 3.1 and 3.4.

Corollary 3.8: *For any undirected graph $G = (V,E)$, $D_V(G) = S_e(G) - S_0(G)$.*

Since $D_V(G) > 0$ for any connected loopless graph G, the following is an immediate consequence of Corollary 3.8.

Corollary 3.9: *If G is a connected undirected graph without self-loops then $S_e(G) > S_0(G)$.*

Now the fact that $|\Theta(G)| = S_e(G) + S_0(G)$ clearly implies that $|\Theta(G)|$ is odd if and only if $D_V(G)$ is odd. Indeed, our next theorem characterizes graphs G for which $D_V(G)$ is odd.

Theorem 3.2: *If $G = (V,E)$ is an undirected graph, then $D_V(G)$ is odd if and only if G is a connected bipartite graph.*

Proof: First note that if G is disconnected or has a self-loop (cycle of length 1), then $D_V(G) = 0$. Conversely, if $D_V(G) = 0$ then, by Proposition 2.1, it fol-

lows that G is either disconnected or has a self-loop. Hence, assume that G is connected and loopless. We proceed by induction on $|E|$.

For the basis step, suppose that $|E| = 0$. Since G is connected, we must have $|V| = 1$. In this case G has no odd cycles with $D_V(G) = 1$ and the basis step is established. For the inductive step, let $G = (V,E)$ be a connected loopless graph with $|E| > 0$ and assume that the theorem holds for all graphs with fewer edges than G. If G is a tree, then it has no odd cycles and $D_V(G) = 1$. Hence, assume that G has at least one cycle. Since G has no self-loops, we can pick any arbitrary edge $x \in E$, and by Corollary 3.5 we have $D_V(G) = D_{V|x}(G|x) + D_V(G - x)$.

Suppose that G has no cycles of odd length, then G necessarily has an even cycle and let x be an edge on such a cycle. Clearly $G|x$ is connected but has an odd cycle. Since $G|x$ has one fewer edge than G, by induction hypothesis $D_{V|x}(G|X)$ is even. On the other hand, $G - x$ is connected and has no odd cycles. Since $G - x$ has $|E| - 1$ edges, $D_V(G - x)$ is odd by the hypothesis. Hence, we conclude that $D_V(G)$ is odd whenever G is connected and has no odd cycles.

Conversely, if G has a cycle of odd length, then we claim that $D_V(G)$ is even. First if G is unicyclic with an odd cycle, then $D_V(G - x)$ and $D_{V|x}(G|x)$ are both odd since $G - x$ and $G|x$ have no odd cycles. Finally consider the case where G has more than one cycle. Let c be a cycle of odd length in G. Pick edge x such that x is not on c. Then $G|x$ and $G - x$ have a cycle of odd length (even if x is a chord of c) so that $D_{V|x}(G|x)$ and $D_V(G - x)$ are even. Therefore, $D_V(G)$ is even as well. ∎

Corollary 3.10: *The complete bipartite graph $K_{p,p}$ is the only graph among the simple graphs on $2p$ points and p^2 edges with an odd all-terminal domination. Likewise, $K_{p,p+1}$ is the only graph with $2p + 1$ points and $p(p + 1)$ edges having odd all-terminal domination.*

Proof: By the theorem of Turan [see page 17, 4] every other graph under consideration has a triangle. ∎

In the remainder of this section we deal with directed graphs. The nature of the invariant domination differs strikingly depending on whether G is a graph or a digraph. As noted in Section 2, the assertion that the clutter $\mathfrak{Z}_K(G)_{+x}$ is obtained from $G|x$ if G is undirected, is no longer valid if G is directed. Due to this anomaly the graph version of Corollary 3.2, namely Corollary 3.3, holds only for graphs and not digraphs. Furthermore, $d_K(G)$ generally can assume any integer value and is never zero if G is an undirected K-graph. On the contrary, for digraphs $d_K(G)$ is 0, +1, or −1. Indeed, a surprising fact is that $d_K(G) = \pm 1$ if and only if G is an acyclic K-digraph, and $d_K(G) = 0$, otherwise [9].

Our first result on digraphs G relates $d_K(G)$ either to $d_{K|x}(G|x)$ or to $d_K(G - x)$, depending upon the nature of x.

THEOREM 3.3: *Suppose $G = (V,E)$ is a digraph, $K \subseteq V$, $s \in K$, and $x = (s,u)$ is an edge of G. Then:*

(*i*) *if the* $\text{indeg}(u) > 1$, *then* $d_K(G) = -d_K(G - x)$.

(*ii*) *if the* $\text{indeg}(u) = 1$ *and x lies on some K-tree rooted at s, then* $d_K(G) = d_{K|x}(G|x)$.

PROOF: (i) Suppose $\text{indeg}(u) > 1$, and let $x' \neq x$ be an edge directed into u. First we show that the system $(E - \{x\}, \mathbf{C}_{+x})$ is not coherent and therefore $d(E - \{x\}, \mathbf{C}_{+x}) = 0$. Indeed, if T' is a K-tree rooted at s which includes x' then $(T' - x') + x$ is a rooted tree that includes all K points. By deleting pendant points which are not in K, if necessary, we may reduce T' to a K-tree T which includes x. Thus, $\{T - x\} \subseteq \{T' - x'\}$ and since $\{T' - x'\}$ is a proper subset of T', it follows that x' does not lie in any of the elements of $\mathbf{C}_{+x}$. If no K-tree of G contains x', then no element of $\mathbf{C}_{+x}$ can contain x'. Hence, the result follows from an application of Corollary 3.2.

(ii) Suppose $\text{indeg}(u) = 1$ and x lies on some K-tree rooted at s. In what follows we show that there exists a one-to-one correspondence between the K-trees rooted at s of G and the $K|x$-trees rooted at s of $G|x$ such that $\{T_i: i \in \Gamma\}$ is a formation of G if and only if $\{T_i|x: i \in \Gamma\}$ is a formation of $G|x$.

First we claim that if T is a K-tree of G rooted at s, then $T|x$ is a $K|x$-tree of $G|x$ rooted at s. Clearly the $\text{indeg}(s) = 0$ in $T|x$ for otherwise $\text{indeg}(u) > 1$, and $T|x$ contains all points of $K|x$. Also, the fact that every pendant point in T is a K-point implies that every pendant point in $T|x$ belongs to $K|x$. Furthermore, the fact that every point other than s has indegree equal to 1 in T implies that every point other than s in $T|x$ has indegree equal to 1. Thus, $T|x$ is a rooted tree, rooted at s. Since $T|x$ contains all points of $K|x$, and its pendant points are in $K|x$, $T|x$ is a $K|x$-tree.

Next we show that the correspondence between the K-trees of G and $K|x$-trees of $G|x$ is one to one. To this end suppose that T_1 and T_2 are two K-trees of G, and $T_1|x = T_2|x$. If x lies in both T_1 and T_2 or neither of these trees, then $T_1 = T_2$. On the other hand, if x lies in T_2 but not in T_1 then $u \notin K$ and the $\text{outdeg}(u) > 0$. This is due to the fact that if $u \in K$, since $\text{indeg}(u) = 1$ in G, x lies on every K-tree of G; if $u \in K$ and $\text{outdeg}(u) = 0$ then T_2 is not a K-tree. Thus, from the facts that $u \notin K$ and $\text{outdeg}(u) > 0$, it follows that none of the edges directed out of u can lie in T_1; this implies that none of these edges can lie in T_2 either because of our assumption that $T_1|x = T_2|x$. Since $u \in V - K$ is a pendant point in T_2, we conclude that T_2 is not a K-tree of G. This contradiction verifies the injectivity.

To see that the correspondence is onto, suppose that T' is a $K|x$-tree of $G|x$. If $u \in K$, edge x is intersected into T' to form a tree T as follows:

(a) If T' contains edges directed out of s which are originally directed out of u in G, then add u to T', connect the tail ends of each of these edges to u leaving the heads unchanged, and finally insert $x = (s,u)$.

(b) If T' contains no edges which originally directed out of u in G, then merely add u and insert $x = (s, u)$.

These constructions are illustrated in Figure 2. Clearly in either case $T|x = T'$ and T is a K-tree of G. If $u \notin K$, then the construction used in (a) is required whenever there are edges in T' which originally emanated from u in G. If no such edges exist in T', we observe that T' itself is a K-tree of G, so $T = T'$ serves our purpose.

Finally we prove the contention concerning the formations of G and $G|x$. If $\{T_i: i \in \Gamma\}$ then it is obvious that $\{T_i|x: i \in \Gamma\}$ is a formation of $G|x$. Conversely, if $\{T_i|x: i \in \Gamma\}$ is a formation of $G|x$, then it is easy to see that $\cup_i T_i$ includes all edges $x' \neq x$. If $u \in K$ then, since $\mathrm{indeg}(u) = 1$ in G, every K-tree of G includes x, whence $x \in \cup_i T_i$. On the contrary, if $u \notin K$ then, since $x = (s, u)$ belongs to some K-tree, there is an edge y directed out of u. Since $\cup_i T_i|x$ is a formation of $G|x$, an index $j \in \Gamma$ exists such that $y \in T_j|x$. Hence, $y \in T_j$ and so $x \in T_j$ as well. This concludes the proof of (ii). ∎

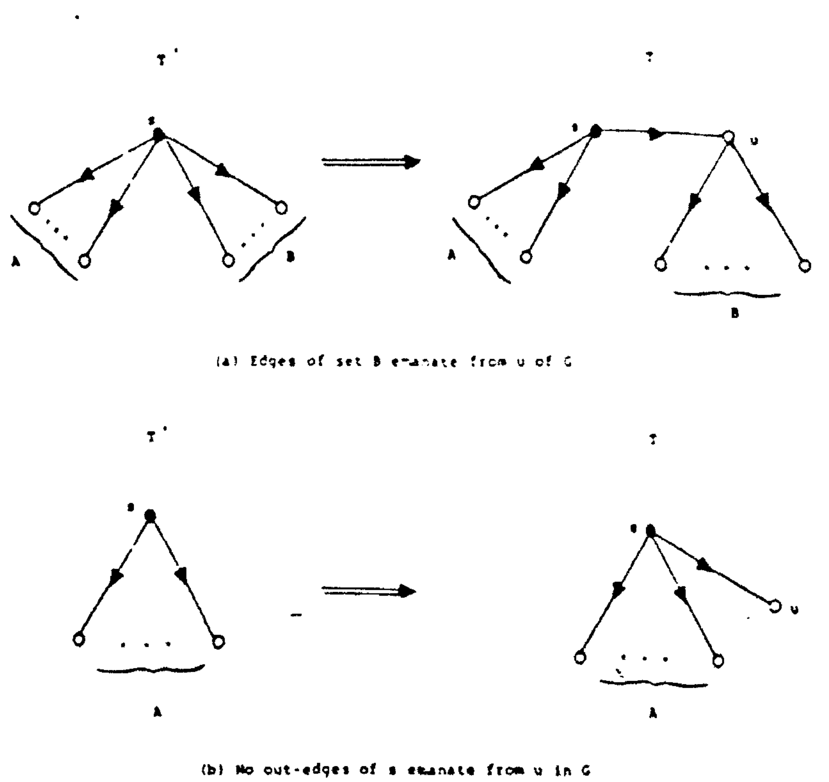


FIGURE 2. An illustration for Theorem 3.3.

Note that $x = (s,u)$ being in some K-tree rooted at s is an essential condition for (ii) to be valid in the above theorem. To see this consider the digraph with two points and one edge $x = (s,u)$, and let $K = \{s\}$.

Theorem 3.3 affords a simple proof of the following well-known result originally established in [9,13].

Corollary 3.11: *Suppose $G = (V,E)$ is a digraph, $K \subseteq V$, $s \in K$. Then $d_K(G) = (-1)^{|E|-|V|+1}$ if G is an acyclic K-graph and $d_K(G) = 0$ otherwise.*

Proof: Since $d_K(G) = 0$ if G is not a K-digraph, we prove the results for acyclic and cyclic K-graphs by induction on $|E|$.

We begin by considering the validity of the result when $|E| = 0$. The second assertion is vacuously true because there are no K-graphs with no edges which also have a directed cycle. The first assertion is trivially true for the solitary K-graph $G = (\{s\},\phi)$ with $K = \{s\}$.

For the inductive step, let $G = (V,E)$ be a K-digraph, $K \subseteq V$, $s \in K$, and $|E| > 0$ and assume that the theorem holds for all K-digraphs with fewer edges than G. Consider an edge $x = (s,u)$ in G.

Suppose indeg$(u) = 1$. If G contains a directed cycle then, since indeg$(s) = 0$ in G, $G|x$ has a directed cycle as well. If $G|x$ is not a $K|x$-graph, then $d_{K|x}(G|x) = 0$. On the other hand, if $G|x$ is a $K|x$-graph then by the induction hypothesis $d_{K|x}(G|x) = 0$. Since indeg$(u) = 1$, it follows from Theorem 3.3, that $d_K(G) = 0$ whenever G is cyclic. Next if G is acyclic then, again by the fact that indeg$(s) = 0$ in G, $G|x$ is also acyclic. Furthermore, it is easy to see that G being an acyclic K-graph implies that $G|x$ is an acyclic $K|x$-graph. Thus, by the induction hypothesis $d_{K|x}(G|x) = (-1)^{(|E|-1)-(|V|-1)+1} = (-1)^{|E|-|V|+1}$. We therefore conclude, by Theorem 3.3, that $d_K(G) = (-1)^{|E|-|V|+1}$ whenever G is an acyclic K-graph.

Suppose indeg$(u) > 1$. If G contains a directed cycle, then $G - x$ must also have a directed cycle. This is due to the fact that indeg$(s) = 0$ in G and so x can not lie on a directed cycle in G. Now if $G - x$ is not a K-graph then $d_K(G - x) = 0$; on the contrary, if $G - x$ is a K-graph, since it is cyclic, then also $d_K(G) = 0$ by the induction hypothesis. Hence, it follows, by Theorem 3.3, that $d_K(G) = 0$ whenever G is a cyclic K-graph. Finally, if G is an acyclic K-graph then we claim that $G - x$ is one as well. That $G - x$ is an acyclic K-graph follows from the observation that if T is a K-tree rooted at s in an acyclic K-graph, then for any edge $x' \neq x$ directed into u, either $(T - x) + x'$ is a K-tree or it can be extended to a K-tree by adding the missing edges to establish a directed path from s to the tail end of x'. Thus, by the induction hypothesis, $d_K(G - x) = (-1)^{(|E|-1)-|V|+1} = -(-1)^{|E|-|V|+1}$. By Theorem 3.3 we may conclude that $d_K(G) = (-1)^{|E|-|V|+1}$ whenever G is an acyclic K-graph, and the proof is complete. ∎

Corollary 3.11 was first established for the case of $K = \{s,t\}$ by Satyanarayana and Prabhakar [13], and later extended to general K in Satyanarayana [9].

We conclude this section with a result analogous to that of Corollary 3.8. Specifically, suppose $G = (V, E)$ is a K-digraph, $K \subseteq V$, and $s \in K$. Let $S^K(G)$ denote the collection of K-subgraphs of G. If $H \in S^K(G)$ and $n(H)$ and $e(H)$ denote the number of points and edges of H, respectively, then $S_0^K(G) = \{H \in S^K(G)$: $(e(H) - n(H) + 1)$ is odd$\}$ and $S_e^K(G) = \{H \in S^K(G)$: $(e(H) - n(H) + 1)$ is even$\}$.

THEOREM 3.4: *If $G = (V, E)$ is an acyclic K-digraph, $K \subseteq V$, $s \in K$, then $d_K(G) = (-1)^{|E|-|V|+1}(|S_e^K(G)| - |S_0^K(G)|)$ and $D_K(G) = |S_e^K(G)| - |S_0^K(G)|$.*

PROOF: We proceed by induction on $|V|$. If $|V| = 1$ then there is only one acyclic K-graph, the one-point tree, for which the theorem is trivially true.

For the inductive step, let $G = (V, E)$ be an acyclic K-digraph with $|V| > 1$, $K \subseteq V$, $s \in K$ and assume that the theorem holds for all acyclic K-graphs with fewer points than G. We now do the secondary induction on $|E|$. Since the underlying undirected graph of G is connected, $|E| \geq |V| - 1$. If $|E| = |V| - 1$, then G must be identical to the only K-tree rooted at s of G. Thus, $|S_e^K(G)| = 1$, $|S_0^K(G)| = 0$ and the theorem follows. For the secondary inductive step, let $G = (V, E)$ such that $|E| > |V| - 1$ and assume that the theorem holds for all acyclic K-graphs with $|V|$ points and fewer than $|E|$ edges. Since G is acyclic, either there are two parallel edges $x = (s, u)$ and $y = (s, u)$ such that $x \neq y$ or there is an edge $x = (s, u)$ such that $\text{indeg}(u) = 1$ in G.

We begin with the first case. In this case, clearly $G - x$ is an acyclic K-graph with one less edge than G, and therefore by the secondary induction hypothesis, $d_K(G - x) = (1)^{(|E|-1)-|V|+1}(|S_e^K(G - x)| - |S_0^K(G - x)|)$ and $D_K(G - x) = |S_e^K(G - x)| - |S_0^K(G - x)|$. Now, using the notation $S^z(H)$ and $S^{z'}(H)$ to denote the K-subgraphs of H containing z and not containing z, respectively, and $S_e^z(H)$, $S_0^z(H)$, $S_e^{z'}(H)$, $S_0^{z'}(H)$ to denote the subclasses of even and odd nullity, we may establish one-to-one correspondences as follows:

$$S_e^{xy}(G) \leftrightarrow S_0^{y}(G - x)$$

$$S_e^{xy'}(G) \leftrightarrow S_e^{y'}(G - x)$$

$$S_e^{x'y}(G) \leftrightarrow S_e^{y}(G - x)$$

$$S_e^{x'y'}(G) \leftrightarrow S_e^{y'}(G - x)$$

and

$$S_0^{xy}(G) \leftrightarrow S_e^{y}(G - x)$$

$$S_0^{xy'}(G) \leftrightarrow S_0^{y'}(G - x)$$

$$S_0^{x'y}(G) \leftrightarrow S_0^{y}(G - x)$$

$$S_0^{x'y'}(G) \leftrightarrow S_0^{y'}(G - x).$$

Now algebraic manipulation yields $|S_e^K(G)| - |S_0^K(G)| = |S_e^K(G - x)| - |S_0^K(G) - x)|$. But then, by Theorem 3.3,

$$d_K(G) = -d_K(G-x) = -(-1)^{(|E|-1)-|V|+1}(|S_e^K(G-x)| - |S_0^K(G-x)|)$$
$$= (-1)^{|E|-|V|+1}(|S_e^K(G)| - |S_0^K(G)|), \quad \text{and}$$
$$D_K(G) = D_K(G-x) = |S_e^K(G-x)| - |S_0^K(G-x)| = |S_e^K(G)| - |S_0^K(G)|.$$

In the second case, where there is an edge $x = (s,u)$ with $\mathrm{indeg}(u) = 1$, we establish a one-to-one correspondence between the K-subgraphs of G and $G|x$ as follows.

The fact that $g|x$ is a $K|x$-subgraph of $G|x$ if and only if g is a K-subgraph of G follows because in this instance, from the proof of Theorem 3.3, $T|x$ is a $K|x$-tree of $G|x$ if and only if T is a K-tree of G. Clearly, the nullities are preserved by this correspondence and so the result follows from the equality $d_K(G) = d_{K|x}(G|x)$. ■

Theorem 3.4 does not hold for cyclic K-graphs. For example consider the digraph G shown of Figure 3 in which $K = \{s,u,v\}$. Clearly $S_0^K(G) = 2$, $S_e^K(G) = 4$, and $d_K(G) = 0$. It is to be noted that, while Theorem 3.4 holds for acyclic K-digraphs where K is arbitrary, Corollary 3.8 holds for undirected graphs only if $K = V$.

4. DOMINATION AND SPANNING TREES

The all-terminal domination $D_V(G)$ of an undirected graph G is, by definition, related to the spanning trees of G since the K-trees of G in this case are the spanning trees. However, if $K \neq V$ not all K-trees are spanning; thus there is no obvious connection between $D_K(G)$ and the spanning trees of G. In this section we show, in fact, that $D_K(G)$, for any arbitrary K, is equal to the number of spanning trees of a certain type. First we require some preliminaries culminating in the central notion of this section.

Suppose $G = (V,E)$ is an undirected graph and $<$ is a strict linear order on E. Let $T = (V,E')$ be a spanning tree of G and $x \in E'$. Then the forest $T - x$

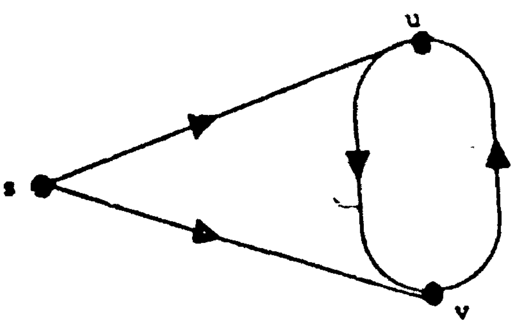


FIGURE 3. An example cyclic digraph.

has exactly two connected components with points sets, say U and $V - U$. The collection of edges of G with one endpoint in U and the other in $V - U$ is called the *fundamental cut determined by x with respect to T*. Likewise, if $x \in \{E - E'\}$ is an edge, then $T + x$ is unicyclic and the cycle in $T + x$ is called the *fundamental cycle determined by x with respect to T*. An edge $x \in E'$ is *internally active in T* if $x < y$ for all $y \in C - x$, where C is the fundamental cut determined by x with respect to T. Finally, an edge $x \notin E'$ is *externally active relative to T* if $x < y$ for all $y \in C - x$, where C is the fundamental cycle determined by x with respect to T. The path of T obtained from the fundamental cycle determined by an externally active edge x is called a *broken cycle* of G [17].

Note that if a spanning tree T has i internally active and j externally active edges, then $0 \leq i \leq |V| - 1$ and $0 \leq j \leq |E| - |V| + 1$.

By $\Upsilon(G)$ we mean the set of all spanning trees of G, while $\Upsilon_{ij}(G)$ denotes the subcollection of $\Upsilon(G)$ of trees having i internally active and j externally active edges. Furthermore, $t(G)$ and $t_{ij}(G)$ denote the cardinalities of $\Upsilon(G)$ and $\Upsilon_{ij}(G)$, respectively. The following is needed for our next definition.

Property K: *If $G = (V,E)$ is an undirected graph and $K \subseteq V$ is a specified subset, then each spanning tree T of G contains exactly one K-tree T^K of G.*

Proof: Each spanning tree which is not a K-tree may be reduced to a K-tree by repeatedly pruning those leaves which are not K-points. ■

Let $\Upsilon_{*0}(G,K)$ consist of those trees $T \in \Upsilon(G)$ satisfying the following conditions:

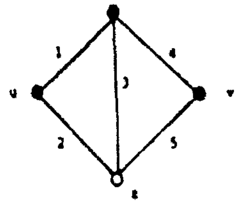
(i) T has no externally active edges, and

(ii) if x is an internally active edge in T, then x is an edge of the unique K-tree T^K contained in T.

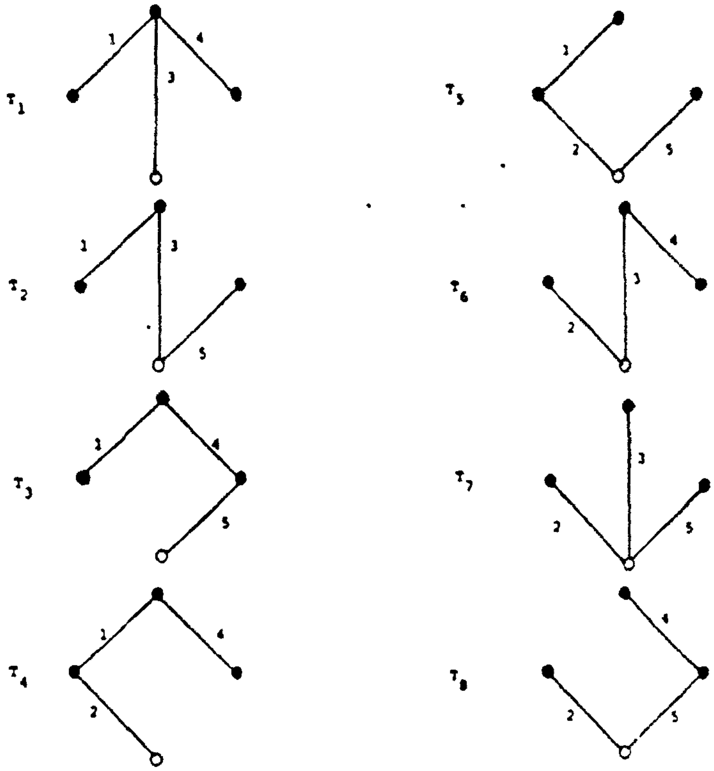
Finally, let $t_{*0}(G,K)$ denote the cardinality of $\Upsilon_{*0}(G,K)$. For example, consider the labeled graph $G = (V,E)$ shown in Figure 4, where the edge labels constitute a strict linear order $<$. Then $\Upsilon(G)$ consists of the eight spanning trees T_1 through T_8 shown in Figure 4, and it follows from the table of Figure 4 that $\Upsilon_{01}(G) = \{T_7\}$, $\Upsilon_{20}(G) = \{T_1, T_5\}$, $\Upsilon_{12}(G) = \phi$, etc. Finally, if $K = \{u,v,w\}$, then $\Upsilon_{*0}(G,K) = \{T_1, T_2, T_5\}$ and $t_{*0}(G,K) = 3$.

Theorem 4.1: *Let $G = (V,E)$, a connected undirected graph with a nonempty subset $K \subseteq V$, and let $<$ be a strict linear order on E. Then $D_K(G) = t_{*0}(G,K)$.*

Proof: Clearly, if G has a self-loop, then $D_K(G) = 0$ and $t_{*0}(G,K) = 0$. We claim that, if G has an edge x with an endpoint $u \in V - K$ such that u is a degree-one point in G, then $D_K(G) = t_{*0}(G,K) = 0$. In this case x is in no K-tree of G, which implies that G has no formations and $D_K(G) = 0$. Since u is a degree-one point of G, x lies in every spanning tree. It is easy to see that x is also internally active in every spanning tree, and since it does not belong to any



(a) Graph $G=(V,E)$, $K = \{u,v,w\}$,
with a linear order $<$ on E.



(b) The spanning trees of G

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
number of internally active edges	2	1	1	3	2	1	0	0
number of externally active edges	0	0	1	0	0	1	1	2

(c) Table showing the internal and external activities of the spanning trees.

FIGURE 4. Illustration for internal and external activities.

K-tree of G, it follows that $t_{\bullet 0}(G,K) = 0$. Hence, in the remainder of the proof, we assume that G is loopless and contains no edges x with an endpoint $u \in V - K$ such that u is a degree-one point in G.

The proof is by induction, first on $|V|$ and then on $|E|$. The result is obvious wher $|V| = 1$. For the primary inductive step, let $G = (V,E)$ be a connected graph with $|V| > 1$, $K \subseteq V$ a nonempty subset, and assume that the theorem holds for all graphs with fewer points than G. Since G is connected, $|E| \geq |V| - 1$. If $|E| = |V| - 1$ then, as G has no degree-one points in $V - K$, G must be a K-tree and we have $D_K(G) = t_{\bullet 0}(G,K) = 1$. Thus, for the secondary inductive step, let $G = (V,E)$ be a graph with $|E| > |V| - 1$, and assume that the theorem holds for all graphs on $|V|$ points and fewer edges than $|E|$.

Let x be the largest edge in the linear order $<$ of G. There are two cases to consider; namely when x is or is not a bridge of G. Suppose x is a bridge of G so that x belongs to every spanning tree of G. Then, if G has a K-tree which does not include x it follows that one of the components of $G - x$ is devoid of K-points. But then $D_K(G) = 0$ and, since x is internally active in every spanning tree, $t_{\bullet 0}(G,K) = 0$ as well. If every K-tree of G contains x, then the correspondence between $T_{\bullet 0}(G,K)$ and $T_{\bullet 0}(G|x,K|x)$ is one to one and onto. Thus, $t_{\bullet 0}(G,K) = t_{\bullet 0}(G|x,K|x)$. But $D_K(G) = D_{K|x}(G|x)$ and the result follows by the primary induction hypothesis.

Next suppose that x is not a bridge of G so that, using Corollary 3.5, we may write that $D_K(G) = D_K(G - x) + D_{K|x}(G|x)$. Utilizing both induction hypotheses, we obtain that $D_K(G) = t_{\bullet 0}(G - x,K) + t_{\bullet 0}(G|x,K|x)$. Hence, it remains to show that $t_{\bullet 0}(G,K) = t_{\bullet 0}(G - x,K) + t_{\bullet 0}(G|x,K|x)$.

Since x is not a bridge and has the largest value in the linear order $<$, it cannot be internally active in any spanning tree. Thus, if T is a spanning tree of G which contains x, the internally active edges of T are the same as those of $T|x$ in $G|x$. Furthermore, the unique K-tree T^K contained in T is either the $K|x$-tree $T|x^K$ contained in $T|x$ or is obtained from it by the addition of x. Thus, every internally active edge of T lies in T^K if and only if every internally active edge of $T|x$ lies in $T|x^K$. Hence, the correspondence between the elements of $T_{\bullet 0}(G,K)$ which include x and the elements of $T_{\bullet 0}(G|x,K|x)$ is one to one and onto. Next observe that T is a spanning tree of G not containing x if and only if T is a spanning tree of $G - x$. Clearly, if an edge y is internally active in T, considered as a spanning tree of G, it is also internally active in T where T is considered as a spanning tree of $G - x$. Conversely, suppose y is internally active in T of $G|x$. Since x is larger than y in $<$, y will remain internally active in T when T is considered as a spanning tree of G. hence, the identity map establishes a one-to-one correspondence between those elements of $T_{\bullet 0}(G,K)$ which do not include x and the elements of $T_{\bullet 0}(G - x,K)$. This concludes the proof. ■

COROLLARY 4.1: *Let $G = (V,E)$ be an undirected graph with a strict linear order $<$ on E. If $x = \{u,v\}$ is the smallest edge of G, then $D_{\{u,v\}}(G) = t_{10}(G)$,*

where $t_{10}(G)$ is the number of spanning trees of G having exactly one internally active edge and zero externally active edges.

PROOF: If G has no edges the result is obvious; otherwise choose the smallest edge x in the linear order $<$. Now if T is a spanning tree with no externally active edges, then x must lie in T and the unique $\{u,v\}$-tree of G contained in T is just x. Thus, the elements of $\mathrm{T}_{\bullet 0}(G,\{u,v\})$ are precisely those spanning trees of G having x as the only internally active edge and no externally active edges. Thus, $t_{\bullet 0}(G,\{u,v\}) = t_{10}(G,\{u,v\}) = t_{10}(G)$. ∎

Let $\mathrm{T}_0(G)$ be the collection of spanning trees having no externally active edges in a graph $G = (V,E)$ with respect to a strict linear order $<$ on E and let $t_0(G) = |\mathrm{T}_0(G)|$. The following is an immediate consequence of Theorem 4.1.

COROLLARY 4.2: *For an undirected graph $G = (V,E)$, $D_V(G) = t_0(G)$.*

It is an obvious consequence of these results that the quantities $t_{\bullet 0}(G,K)$, and $t_0(G)$ are invariant with respect to the linear order $<$ of the edges of G. Indeed, Tutte [16], in his study of the chromatic polynomial of a graph, has noted this fact for all parameters $t_{ij}(G)$. The value of the chromatic polynomial, $P(G;\lambda)$ of a graph G gives the number of proper λ-colorings of G; that is, the number of ways of assigning colors to the points of G, using λ or fewer colors, so that no two adjacent points are assigned the same color. Tutte [16] showed that for any connected graph G, $|(P(G;\lambda)/(1-\lambda))|_{\lambda=1}| = t_{10}(G) = t_{01}(G)$. Hence, the next result follows directly from Corollary 4.1.

COROLLARY 4.3: *For any connected undirected graph $G = (V,E)$ and any edge $x = (u,v)$ such that $u \neq v$, $D_{\{u,v\}}(G) = |(P(G;\lambda)/(1-\lambda))|_{\lambda=1}|$.*

An immediate consequence of this corollary is that $D_x(G) = D_y(G)$ for any pair of edges x and y of G. Moreover, Whitney [17] showed that $P(G;\lambda) = \sum_{i=1}^{|V|} (-1)^{|V|-i} m_i(G)\lambda^i$ where $m_i(G)$ is the number of spanning forests of G with i connected components and having no externally active edges. Note that an edge is externally active with respect to a given forest if and only if it is externally active with respect to some tree of the forest. Clearly when $i = 1$, then the spanning forests are the spanning trees of G; whence $m_1(G) = t_0(G)$. We therefore have the following corollary.

COROLLARY 4.4: *If $G = (V,E)$ is an undirected graph, then $|(P(G;\lambda)/\lambda)|_{\lambda=0}| = D_V(G)$.*

In a recent work, Satyanarayana and Tindell [15] introduced a polynomial $P(G,K;\lambda)$ in λ determined by graph $G = (V,E)$, $K \subseteq V$. Like the classical chromatic polynomial $P(G;\lambda)$, this new polynomial has integer coefficients that alternate in sign. Furthermore $P(G,K;\lambda) = P(G;\lambda)$ if K is the entire point set of G. This new polynomial has several interesting properties, and in particular, it has been shown that $|(P(G,K;\lambda)/\lambda)|_{\lambda=0}| = D_K(G)$.

Another interesting connection between $D_K(G)$ and the number of certain orientations of G was discovered by Satyanarayana and Procesi-Ciampi [14]. An *orientation* of an undirected graph is an assignment of direction to each edge of the graph. Let $G = (V,E)$ be a connected undirected graph, and suppose $K \subseteq V$. A *rooted orientation*, with respect to K and the root $s \in K$, of G is an orientation of G such that exactly one point of the orientation, namely s, has indegree = 0 and every point of outdegree = 0 belongs to K. An orientation is *acyclic* if it has no directed cycles and is *cyclic* otherwise. The result proved in Satyanarayana and Procesi-Ciampi [14] asserts that if $N_K(G,s)$ is the number of rooted acyclic orientations of G, with respect to K and the root $s \in K$, then $D_K(G) = N_K(G,s)$ for all $s \in K$. An immediate consequence of this result is the fact that, if $i \in K$ and $j \in K$ are two points of G, then $N_K(G,i) = N_K(G,j)$ and hence the number of rooted acyclic orientations of a graph, with respect to a given K, is independent of the root selected from K.

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